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Kontsevich and Takhtajan Construction of Star Product on the Poisson–Lie Group $GL(2)$

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Comparing the star product defined by Takhtajan on the Poisson–Lie group $GL(2)$ and any star product calculated from the Kontsevich's graphs (any "K-star product") on the same group, we show, by direct computation, that the Takhtajan star product on $GL(2)$ can't be written as a K-star product.

KEY WORDS: Poisson–Lie groups; star products; quantum groups.

1. INTRODUCTION

In recent years great progress was made in developing new approach and deriving exact result in deformation of different groups and algebras. Each of these deformation theories is not independent of the others.

In fact, since Kontsevich's well-known preprint (Kontsevich, q-alg/9709040), in which he gives a universal construction of a star product on $\mathbb{R}^d$ endowed with an arbitrary Poisson structure, several authors tempted to bring this approach closer to others already existing, let us cite, for instance, Arnal *et al.* (1999) and Dito (1999) who give by two different manners an equivalence between the Kontsevich and Gutt (1983) star product on the dual of Lie algebra, and Kathotia (q-alg/9811174) and Shoikhet (q-alg/9903036) who related the Kontsevich formula to Campbell–Baker–Hausdorff's one on the dual of Lie algebra.

The starting point of the present idea is the Drinfeld universal approach to construct quantum groups (Drinfeld, 1986). This mathematical structure arises in particular from quantization of some Poisson bracket on "usual" Lie groups obtained from a classical r-matrix satisfying the Yang Baxter Equation. Here we tempt to eliminate the relation between the star product construct by Takhtajan (basing on Drinfeld's work), on the particular Lie group $GL(2)$ endowed with

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a certain r-matrix, which satisfies the modified Yang Baxter Equation, and the star product constructed on this Poisson–Lie group from the Kontsevich's graphs ("K-star product") either on $GL(2)$ view as an open subset of $\mathbb{R}^4$ or on the domain of an exponential chart near the origin. By a direct computation, we show that the Takhtajan star product cannot be written as a K-star product.

This paper is organized as follows. The second section is devoted to a review of basic definitions of the quantization of Poisson–Lie group. The third section introduces the Kontsevich construction, in Section 4 we give a generalization of this construction. Then we give the quantization of the particular Poisson–Lie group $GL(2)$ in the fifth section. Finally, we get our main result by comparing the two star product on an "ordinary" and an "exponential" chart in the three last sections.

2. USUAL QUANTIZATION OF POISSON–LIE GROUP

Let us first recall the aim of the construction of quantum groups by Drinfeld (1986) and Takhtajan (1989). Let G be a Lie group with Lie algebra g, we denote by (X_i) a basis of g and $U(g)$ the universal enveloping algebra of g. If $r \in \Lambda^2 g$, we consider the elements r^{12}, r^{13}, r^{23} of $U(g) \otimes U(g) \otimes U(g)$ definded by

$$r^{12} = r^{ij} X_i \otimes X_j \otimes 1$$
$$r^{13} = r^{ij} X_i \otimes 1 \otimes X_j$$
$$r^{23} = r^{ij} 1 \otimes X_i \otimes X_j$$

where $r = r^{ij} X_i \otimes X_j$. We say that r satisfies the modified Classical Yang-Baxter Equation (CYBE) if

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = I_{123}, \quad I_{123} \in \Lambda^3 g \tag{1}$$

and

$$[I_{123}, 1 \otimes 1 \otimes X + 1 \otimes X \otimes 1 + X \otimes 1 \otimes 1] = 0 \quad \forall X \in g \tag{2}$$

(here the bracket is the commutator in the associative algebra $U(g) \otimes U(g) \otimes U(g)$). Such an element is called a r-matrix.

To each r, we associate a Poisson structure on G by putting

$$\{\varphi, \psi\} = r^{ij} \left[X_i^\ell(\varphi) X_j^\ell(\psi) - X_i^r(\varphi) X_j^r(\psi)\right] \quad \varphi, \psi \in C^\infty(G) \tag{3}$$

where X_i^ℓ(resp. X_j^r) are the left-invariant (resp. right-invariant) vector fields on G corresponding to X_i(resp. X_j).

Definition 1 (Poisson–Lie group). A Poisson–Lie group is a Lie group G endowed with a Poisson structure $\{\,,\,\}$ associated to a r-matrix satisfying the modified CYBE.

The quantization of a Poisson–Lie group $(G, \{\,,\,\})$ is a deformation of the commutative algebra $C^\infty(G)$ which turns it to a new noncommutative algebra $C^\infty(G)[\![t]\!]$, where t is a deformation parameter. The algebra $C^\infty(G)[\![t]\!]$ as a vector space coincides with $C^\infty(G)$, but has a new product $*$ called a star product.

Definition 2 (Star product). A star product on a Poisson manifold is a map:

$$* : C^\infty(G) \otimes C^\infty(G) \to C^\infty(G)[\![t]\!]$$
$$\varphi * \psi = \varphi \cdot \psi + \sum_{i=1} C_i(\varphi, \psi) t^i$$

such that, for all $\varphi, \psi, \chi \in C^\infty(G)$

(1) C_i is a bidifferential operator on $C^\infty(G)$,
(2) $\varphi * 1 = 1 * \varphi = \varphi$,
(3) $\{\varphi, \psi\} = \lim_{t\to 0} \frac{1}{t}(\varphi * \psi - \psi * \varphi)$, and
(4) $(\varphi * \psi) * \chi = \varphi * (\psi * \chi)$.

Since G is a group, there is a naturel comultiplication Δ on $C^\infty(G)$:

$$\Delta(\varphi)(x, y) = \varphi(xy) \quad (\varphi \in C^\infty(G), x, y \in G).$$

A star product preserving Δ, i.e., such that

$$\Delta(\varphi * \psi) = \Delta(\varphi) * \Delta(\psi) \tag{4}$$

where $*$ is naturally extended to $C^\infty(G) \otimes C^\infty(G)$ and was built by Drinfeld (1986) and Takhtajan (1989) in a purely algebraic way. They first look for a formal element $F \in U(g) \otimes U(g)[\![t]\!]$ such that the product

$$\varphi * \psi = (F^{-1})^r (F)^\ell (\varphi \otimes \psi) \tag{5}$$

is a star product. And the associativity axiom looks

$$F(X + Y, Z)F(X, Y) = \alpha(X, Y, Z)F(X, Y + Z)F(Y, Z) \tag{6}$$

where $\alpha(X, Y, Z) \in U(g) \otimes U(g) \otimes U(g)[\![t]\!]$ is G-invariant:

$$(\alpha, 1 \otimes 1 \otimes X + 1 \otimes X \otimes 1 + X \otimes 1 \otimes 1) = 0 \quad \forall X \in g. \tag{7}$$

In order to have this, we need that

$$F = 1 - \frac{t}{2} r + \sum_{n\geq 2} F_n t^n$$

and

$$F(X, 0) = F(0, Y) = 1;$$

this implies that α has the following form:

$$\alpha = 1 + t^2 \alpha_2 + \cdots$$

with

$$\mathrm{Alt}(\alpha_2) = -4I_{123}.$$

Here Alt stands for the alternation, i.e.,

$$\begin{aligned}\mathrm{Alt}(\alpha_2)(X, Y, Z) = \alpha_2(X, Y, Z) - \alpha_2(Y, X, Z) + \alpha_2(Y, Z, X)\\ - \alpha_2(Z, Y, X) + \alpha_2(Z, X, Y) - \alpha_2(X, Z, Y)\end{aligned}$$

and

$$\alpha(X, Y, Z)\alpha(X, Y + Z, U)\alpha(Y, Z, U) = \alpha(X + Y, Z, U)\alpha(X, Y, Z + U).$$

An explicit solution for $GL(2)$ will be given later.

3. KONTSEVICH'S STAR PRODUCT ON $\mathbb{R}^d$

In order to construct a star product on any Poisson manifold, M. Kontsevich built first such a star product for any Poisson structure Λ on a flat space $\mathbb{R}^d$ with a given system of coordinates.

He considers a set $G_{n,m}$ of graphs Γ with two kinds of vertices: n aerial vertices $p_1, p_2, \ldots, p_n$ and m terrestrial vertices $q_1 < q_2 < \cdots < q_m$. From each aerial vertex p_i, two edges (arrows) $\vec{a}_i$ are starting, they end at any different vertices (a) distincts from p_i (i.e. there are not parallel multiple edges either "small" loop); on the edges, we fix the lexicographic ordering, we associate to the graph Γ an m-differential operator:

$$\begin{aligned}&B_\Gamma(\Lambda \otimes \Lambda \otimes \Lambda)(\varphi_1, \varphi_2, \ldots, \varphi_m)\\ &\quad = \sum D_{p_1}\Lambda_1^{i_1 i_2 \cdots i_{k_1}} \cdots D_{p_n}\Lambda_n^{i_{k_1} + \cdots k_{n-1} + 1 \cdots i_{k_1} + \cdots k_n} D_{q_1}\varphi_1 \cdots D_{q_m}\varphi_m \qquad (8)\end{aligned}$$

where D_a is the operator:

$$D_a = \prod_{l, \mathrm{edge}(l) = .\vec{a}} \partial_{i_l}.$$

Kontsevich looks for a star product of the form

$$\varphi * \psi = \varphi \cdot \psi + \sum_{n \geq 1} t^n \sum_{\Gamma \in G_{n,2}} a_\Gamma B_\Gamma(\Lambda, \Lambda, \ldots, \Lambda)(\varphi \otimes \psi) \qquad (9)$$

where a_Γ is a constant. An explicit universal choice of the a_Γ is given by Kontsevich (q-alg/9709040), a_Γ is the integral of a certain form ω_Γ defined from Γ on a configuration space $C_{n,2}^+$. With this choice for any Poisson structure Λ, the star product of Kontsevich satisfies the conditions (1), (2), (3), and (4) of the preceding definition.

4. GENERALIZATION OF THE KONTSEVICH CONSTRUCTION

We first generalize the construction of Kontsevich on $\mathbb{R}^d$. Let us consider now graphs with n aerial vertices $p_1, p_2, \ldots, p_m$ and m terrestrial vertices $q_1, q_2, \ldots, q_m$ and two edges starting from each aerial vertex p_i and ending at any vertex (even possibly in p_i itself) without any double edge.

Since we need property (2) of Definition 2 for our star product, we restrict ourselves to graphs for which $B_\Gamma(1, \varphi) = B_\Gamma(\varphi, 1) = 0$, i.e., to graphs such that, for any terrestrial vertex q_j, at least one edge is ending. Let us denote by $\tilde{G}_{n,m}$ the set of such graphs.

Definition 3 (K-star product). A K-star product on $\mathbb{R}^d$ is a star product of the form:

$$\varphi * \psi = \varphi \cdot \psi + \sum_{n\geq 1} t^n \sum_{\Gamma \in \tilde{G}_{n,2}} a_\Gamma B_\Gamma(\Lambda, \Lambda, \ldots, \Lambda)(\varphi \otimes \psi)$$

where a_Γ is a constant.

Remark. Kontsevich needs to eliminate the "small loops" $p_i \vec{p}_i$ in order to define the form ω_Γ, but he considers such a generalization for linear Λ in Kontsevich (q-alg/9709040).

Up to the ordering of the aerial vertices, it is easy to consider all elements of $\tilde{G}_{2,2}$:

Lemma (Description of $\tilde{G}_{2,2}$). *The set $\tilde{G}_{2,2}$ contains exactly 10 graphs. (See Fig. 1.)*

If we restrict ourselves to a symmetric C_2 in our star product, we have only to consider six graphs or linear combination of graphs (see Fig. 2).

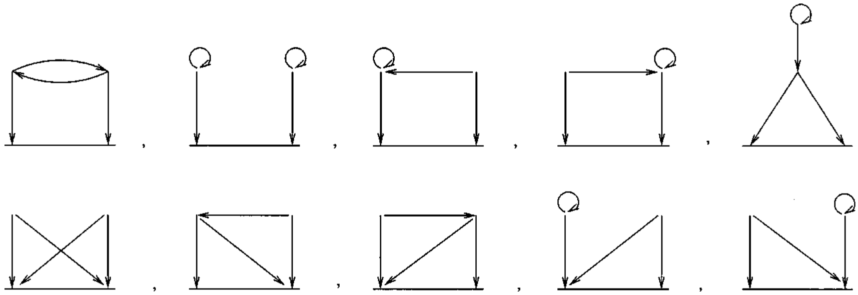


Fig. 1. The $\tilde{G}_{2,2}$ elements.

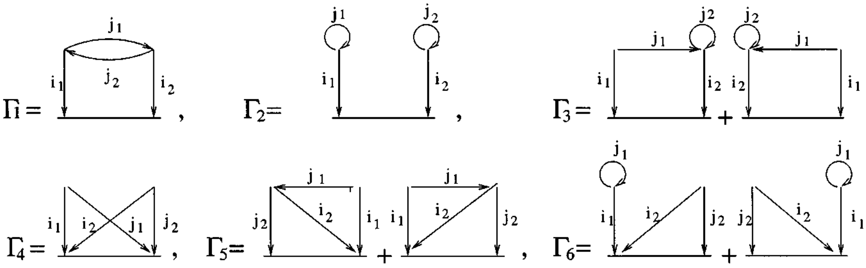


Fig. 2. The symmetric elements of $\tilde{G}_{2,2}$.

5. THE QUANTUM GROUP $GL(2)$

Let us now consider the particular case of Lie group $G = GL(2) \subset \mathbb{R}^4$. We endow $GL(2)$ with a Poisson–Lie structure by defining a r-matrix $\tilde{r}$, which verifies the modified CYBE:

$$\tilde{r} = X_+ \otimes X_- - X_- \otimes X_+ \quad \in \Lambda^2 g$$

where

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

So the corresponding Poisson bracket on $GL(2)$ has the following form:

$$\{\varphi, \psi\} = X_+^\ell(\varphi)X_-^\ell(\psi) - X_-^\ell(\varphi)X_+^\ell(\psi) - X_+^r(\varphi)X_-^r(\psi) + X_-^r(\varphi)X_+^r(\psi).$$

We consider the matrix $T = (t_{ij})_{i,j=1,2}$ of coordinate functions on $GL(2)$, i.e., the functions $t_{ij}(g) = g_{ij}$, where, for $g \in G$, we denote by g_{ij} its matrix elements. Let us put

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Left and Right actions of G on matrix coordinates on G are given by

$$\begin{aligned} \left(X^\ell t_{ij}\right)(g) &= (gX)_{ij} = \sum_k t_{ik}(g)X_{kj} \\ \left(X^r t_{ij}\right)(g) &= (Xg)_{ij} = \sum_k X_{ik}t_{kj}(g) \end{aligned} \tag{10}$$

with these notations, the Poisson bracket looks like

$$\begin{aligned} &\Lambda^{ab} = \{a, b\} = ab,\ \Lambda^{ac} = \{a, c\} = ac,\ \Lambda^{bc} = \{b, c\} = 0, \\ &\Lambda^{bd} = \{b, d\} = bd,\ \Lambda^{cd} = \{c, d\} = cd,\ \Lambda^{ad} = \{a, d\} = 2bc. \end{aligned}$$

These relations define completely the Poisson–Lie group $GL(2)$ with r-matrix $\tilde{r}$ since any $\varphi \in C^\infty(G)$ can be approximated by polynomial functions in a, b, c, d.

Now, what about the quantization of this Poisson–Lie group, i.e., how looks the star product in terms of coordinate functions? Takhtajan (1989) gives an elegant form of his star product (5):

$$T_1 * T_2 = F^{-1} T \otimes T F \tag{11}$$

with

$$T_1 = T \otimes I$$
$$T_2 = I \otimes T.$$

A solution of Eq. (6) for $GL(2)$ was given by

$$F = e^{\frac{-tP}{2}} \begin{pmatrix} \sqrt{q} & 0 & 0 & 0 \\ 0 & u^{-1} & 0 & 0 \\ 0 & v & u & 0 \\ 0 & 0 & 0 & \sqrt{q} \end{pmatrix} \tag{12}$$

where $q = e^t$, $u = \sqrt{\frac{2}{q+q^{-1}}}$, $v = \frac{q-q^{-1}}{\sqrt{2(q+q^{-1})}}$, and P is the permutation operator.

We shall call the corresponding star product the Takhtajan star product and denote it by $*_T$.

Proposition 1 (Computation of $*_T$) (Takhtajan, 1989). *Taking the form (12) of element F, we obtain the following relations:*

$$a *_T a = a^2,\ b *_T b = b^2,\ c *_T c = c^2,\ d *_T d = d^2$$
$$a *_T b = \sqrt{\frac{2}{1+q^{-2}}} ab,\ a *_T c = \sqrt{\frac{2}{1+q^{-2}}} ac,\ b *_T c = \frac{2}{q+q^{-1}} bc$$
$$b *_T d = \sqrt{\frac{2}{1+q^{-2}}} bd,\ c *_T d = \sqrt{\frac{2}{1+q^{-2}}} cd,\ a *_T d = ad + \frac{q-q^{-1}}{q+q^{-1}} bc. \tag{13}$$

6. COMPARING STAR PRODUCT

We want now to compare the Takhtajan star product and the K-star product. We shall compare these two sorts of star products on a chart domain, i.e., first we look at $GL(2)$ as an open subset of $\mathbb{R}^4$:

$$GL(2) = \left\{ T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc \neq 0 \right\} \subset \mathbb{R}^4 = \{(a, b, c, d)\}$$

we call this chart the “*ordinary chart*.” Then we look at the expontial mapping:

$$\exp : g\ell(2) = \left\{ X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \|X\| < 2\pi \right\} \to \{e^X\} \subset GL(2)$$

we call this chart the “exponential chart.”

The Takhtajan star product $*_T$ can be written on the ordinary or exponential chart as

$$\varphi *_T \psi = \varphi \cdot \psi + tC_1(\varphi, \psi) + t^2 C_T(\varphi, \psi) + \cdots$$

with C_T symmetric.
Suppose that $*_T$ is a K-star product, then C_T has the form

$$C_T = a_{\Gamma_1} B_{\Gamma_1}(\Lambda, \Lambda) + a_{\Gamma_2} B_{\Gamma_2}(\Lambda, \Lambda) + \cdots + a_{\Gamma_6} B_{\Gamma_6}(\Lambda, \Lambda).$$

Computing Eq. (13), we find thus relations between the a_{Γ_i} and it is possible to prove there is no solution for these relations. We shall apply this method for the exponential chart.

Another possible way is to use the graph cohomology (Arnal and Masmoudi, 2002). If we write the Kontsevich star product

$$\varphi *_K \psi = \varphi \cdot \psi + tC_1(\varphi, \psi) + t^2 C_K(\varphi, \psi) + \cdots$$

Suppose that $*_T$ is a K-star product, then $C_K - C_T$ being symmetric is a coboundary δT with $T = \sum_{n=1,2} t^n \sum_{\Gamma \in \tilde{G}_{n,1}} K_\Gamma B_\Gamma(\Lambda, \Lambda)$. We can compute T and prove there is no solution again. We shall apply this method for the ordinary chart.

7. IN THE ORDINARY CHART

On the Poisson–Lie group $GL(2) \subset \mathbb{R}^4$ we consider the chart $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, in this case we have

$$\varphi *_K \psi = \varphi \cdot \psi + tC_1(\varphi, \psi) + t^2 C_K(\varphi, \psi) + \cdots$$

$$\varphi *_T \psi = \varphi \cdot \psi + tC_1(\varphi, \psi) + t^2 C_T(\varphi, \psi) + \cdots$$

where $C_K(C_T)$ is the Kontsevich (the Takhtajan) bidifferential operator.

Since Λ is quadratic, if φ, ψ are coordinate functions, each term of these star products is quadratic. Now C_K and C_T are symmetrics.

If we assume that we can write C_T as

$$C_T = \sum_{\Gamma \in \tilde{G}_{2,2}} a_\Gamma B_\Gamma(\Lambda, \Lambda)$$

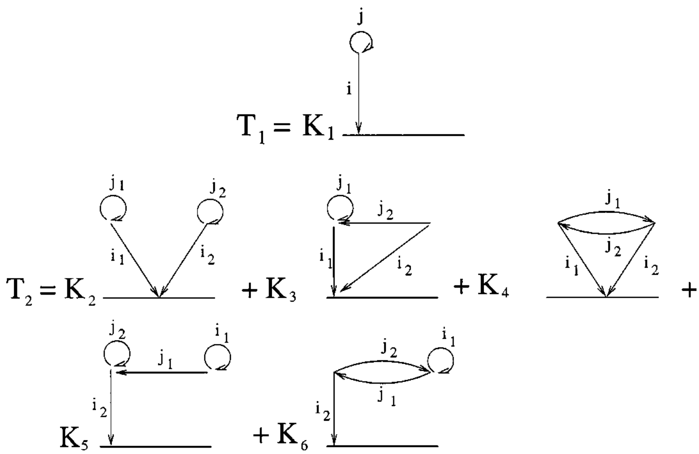
then $C_K - C_T$ is a Hochschild cocycle which is symmetric and vanishing on constants, i.e., a coboundary, and there exists differential operators vanishing on constants T_1, T_2 such that

$$T\varphi = \varphi + tT_1\varphi + t^2T_2\varphi \tag{14}$$

satisfies

$$T(\varphi *_K \psi) = T(\varphi) *_T T(\psi) \tag{15}$$

and then



$$\begin{aligned} T \;=\; & Id + t\sum_{ij} K_1 \partial_j \Lambda^{ij} \partial_i + t^2 \sum_{i_1 i_2 j_1 j_2} K_2 \partial_{j2} \Lambda^{i_2 j_2} \partial_{j_1} \Lambda^{i_1 j_1} \partial_{i_1} \partial_{i_2} \\ & + t^2 \sum_{i_1 i_2 j_1 j_2} K_3 \Lambda^{i_2 j_2} \partial_{j_1} \partial_{j_2} \Lambda^{i_1 j_1} \partial_{i_1} \partial_{i_2} + t^2 \sum_{i_1 i_2 j_1 j_2} K_4 \partial_{j_1} \Lambda^{i_2 j_2} \partial_{j_2} \Lambda^{i_1 j_1} \partial_{i_1} \partial_{i_2} \\ & + t^2 \sum_{i_1 i_2 j_1 j_2} K_5 \partial_{i_1} \Lambda^{i_1 j_1} \partial_{j_2} \partial_{j_1} \Lambda^{i_2 j_2} \partial_{i_2} + t^2 \sum_{i_1 i_2 j_1 j_2} K_6 \partial_{j_2} \partial_{j_1} \Lambda^{i_1 j_1} \partial_{i_1} \Lambda^{i_2 j_2} \partial_{i_2} \end{aligned} \tag{16}$$

with the 4-upple of indexes $(i_1, i_2, j_1, j_2) \in \{a, b, c, d\}^4$. So the equivalence between the two star products $*_T$ and $*_K$

$$\sum_{p+q=2} T_p (C_K)_q (\varphi, \psi) = \sum_{p+q+r=2} (C_T)_p (T_q\varphi, T_r\psi)$$

gives the following system of equations:

$$\begin{cases} 2K_3 + K_4 = \frac{7}{48} \\ K_3 + 2K_4 = \frac{1}{6} \\ K_4 - \frac{1}{12} = 0 \\ K_4 - \frac{1}{12} = -\frac{1}{8} \\ K_1^2 + 2K_2 + K_4 = \frac{1}{12}. \end{cases} \tag{17}$$

This system has no solution.

Proposition 2 (*Comparing on ordinary chart*). *In the ordinary chart the Takhtajan star product can't be written as a K-star product.*

8. IN THE EXPONENTIAL CHART

Let us consider again the Poisson–Lie group $GL(2)$, but now, with an exponential chart $X = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ such that $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = e^X$.

In this case we have

$$\begin{cases} a = 1 + \alpha + \frac{\alpha^2+\beta\gamma}{2} + \frac{\alpha^3+\beta\gamma\alpha+\beta\gamma\delta}{6} + \cdots \\ b = \beta + \frac{\beta\alpha+\beta\delta}{2} + \frac{\beta^2\gamma+\beta\alpha^2+\beta\delta^2+\beta\alpha\delta}{6} + \cdots \\ c = \gamma + \frac{\gamma\alpha+\gamma\delta}{2} + \frac{\beta\gamma^2+\gamma\alpha^2+\gamma\delta^2+\gamma\alpha\delta}{6} + \cdots \\ d = 1 + \delta + \frac{\delta^2+\beta\gamma}{2} + \frac{\delta^3+\beta\gamma\delta+\beta\gamma\alpha}{6} + \cdots \end{cases} \tag{18}$$

and the Poisson structures up to third order looks as

$$\begin{cases} \Lambda^{\alpha\beta} = \beta + \frac{1}{3}\beta^2\gamma + \frac{1}{3}\beta\alpha^2 + \cdots \\ \Lambda^{\alpha\gamma} = \gamma + \frac{1}{3}\beta\gamma^2 + \frac{1}{3}\gamma\alpha^2 + \cdots \\ \Lambda^{\beta\delta} = \beta + \frac{1}{3}\beta^2\gamma + \frac{1}{3}\beta\delta^2 + \cdots \\ \Lambda^{\gamma\delta} = \gamma + \frac{1}{3}\beta\gamma^2 + \frac{1}{3}\gamma\delta^2 + \cdots \\ \Lambda^{\beta\gamma} = 0 \\ \Lambda^{\alpha\delta} = \beta\gamma\alpha + \beta\gamma\delta + \cdots \end{cases} \tag{19}$$

If we try to write the Takhtajan star product as a K-star product, we have to consider all symmetric graphs Γ_1, Γ_2, Γ_3, Γ_4, Γ_5, and Γ_6, described in lemma (Section 4).

We attribute respectly the weights a_{Γ_1}, a_{Γ_2}, a_{Γ_3}, a_{Γ_4}, a_{Γ_5}, and a_{Γ_6} to graphs Γ_1, Γ_2, Γ_3, Γ_4, Γ_5, and Γ_6 such that the product

$$\varphi * \psi = \sum_{n=0} t^n \sum_{\Gamma \in \tilde{G}_{n,2}} a_\Gamma B_\Gamma(\Lambda, \ldots, \Lambda)(\varphi \otimes \psi)$$

is associative.

So we calculate the operator

$$\sum_{i_1 i_2 j_1 j_2} a_{\Gamma_1} \partial_{j_1} \Lambda^{i_2 j_2} \partial_{j_2} \Lambda^{i_1 j_1} \partial_{i_1} \otimes \partial_{i_2} + \sum_{i_1 i_2 j_1 j_2} a_{\Gamma_2} \partial_{j_1} \Lambda^{i_1 j_1} \partial_{j_2} \Lambda^{i_2 j_2} \partial_{i_1} \otimes \partial_{i_2}$$
$$+ \sum_{i_1 i_2 j_1 j_2} a_{\Gamma_3} \Lambda^{i_1 j_1} \partial_{j_1} \partial_{j_2} \Lambda^{i_2 j_2} \left(\partial_{i_1} \otimes \partial_{i_2} + \partial_{i_2} \otimes \partial_{i_1} \right)$$
$$+ \sum_{i_1 i_2 j_1 j_2} a_{\Gamma_4} \Lambda^{i_1 j_1} \Lambda^{i_2 j_2} \partial_{i_2} \partial_{i_1} \otimes \partial_{j_2} \partial_{j_1}$$
$$+ \sum_{i_1 i_2 j_1 j_2} a_{\Gamma_5} \Lambda^{i_1 j_1} \partial_{j_1} \Lambda^{i_2 j_2} \left(\partial_{i_2} \partial_{i_1} \otimes \partial_{j_2} + \partial_{j_2} \otimes \partial_{i_2} \partial_{i_1} \right)$$
$$+ \sum_{i_1 i_2 j_1 j_2} a_{\Gamma_6} \Lambda^{i_2 j_2} \partial_{j_1} \Lambda^{i_1 j_1} \left(\partial_{i_2} \partial_{i_1} \otimes \partial_{j_2} + \partial_{j_2} \otimes \partial_{i_2} \partial_{i_1} \right) \tag{20}$$

associated to the graphs of (Fig. 2), on each pair (φ, ψ) of functions $\varphi, \psi \in \{a, b, c, d\}$ and 4-upple of indexes $(i_1, i_2, j_1, j_2,) \in \{\alpha, \beta, \delta, \gamma\}^4$. Then the vanishing of the bidifferential operator $C_K - C_T$ gives this system of equations:

$$\begin{cases} a_{\Gamma_1} + 2a_{\Gamma_2} = 0 \\ 10a_{\Gamma_1} + 32a_{\Gamma_2} + 28a_{\Gamma_3} - 6a_{\Gamma_4} - 8a_{\Gamma_5} - 16a_{\Gamma_6} = 0 \\ 8a_{\Gamma_3} + a_{\Gamma_5} + 4a_{\Gamma_6} = 0 \\ 8a_{\Gamma_3} + 2a_{\Gamma_5} + 4a_{\Gamma_6} = -\frac{3}{2} \\ a_{\Gamma_5} + 2a_{\Gamma_6} = -\frac{1}{8} \\ -a_{\Gamma_1} - 2a_{\Gamma_2} - 6a_{\Gamma_3} + 4a_{\Gamma_5} + 8a_{\Gamma_6} = -\frac{9}{16} \\ -a_{\Gamma_1} + 2a_{\Gamma_2} + 2a_{\Gamma_3} + 2a_{\Gamma_5} + 4a_{\Gamma_6} = -\frac{3}{16} \end{cases} \tag{21}$$

which is a system with no solution.

For instance let us give the calculation, up to second order, of $C_K(a, d)$ as an example from which we obtain the first and second equations.

We determine $B_{\Gamma_n}(a, d)$ for $n = 1, 2, \ldots, 6$. We get for (see Fig. 3) the functions:

$$B_{\Gamma_1}(a, d) = \sum_{i_1 i_2 j_1 j_2} \partial_{j_1} \Lambda^{i_2 j_2} \partial_{j_2} \Lambda^{i_1 j_1} \partial_{j_1} a \partial_{i_2} d$$

Case 1.

$$\begin{cases} i_1 = \alpha \\ j_1 = \beta \end{cases}$$

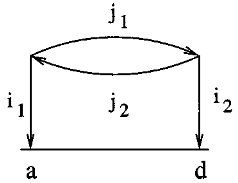


Fig. 3. Γ_1.

So if we calculate $\partial_\beta \Lambda^{i_2 j_2} \partial_{j_2} \Lambda^{\alpha\beta} \partial_\alpha a \partial_{i_2} d$, with $i_2, j_2 \in \{\alpha, \beta, \gamma, \delta\}$, we get

$$\left(\partial_\beta \Lambda^{i_2 j_2} \partial_{j_2} \Lambda^{\alpha\beta} \partial_\alpha a \partial_{i_2} d\right)_{i_2 j_2=\alpha,\beta,\gamma,\delta} = -1 - \alpha - \delta - \alpha\delta - \frac{5}{6}\alpha^2 - \frac{5}{6}\delta^2 - \frac{11}{6}\beta\gamma + 0(2)$$

Case 2.

$$\begin{cases} i_1 = \beta \\ j_1 = \alpha \end{cases} \quad \left(\partial_\alpha \Lambda^{i_2 j_2} \partial_{j_2} \Lambda^{\beta\alpha} \partial_\beta a \partial_{i_2} d\right)_{i_2 j_2=\alpha,\beta,\gamma,\delta} = 0(2)$$

Case 3.

$$\begin{cases} i_1 = \alpha & \left(\partial_\gamma \Lambda^{i_2 j_2} \partial_{j_2} \Lambda^{\alpha\gamma} \partial_\alpha a \partial_{i_2} d\right)_{i_2 j_2=\alpha,\beta,\gamma,\delta} \\ j_1 = \gamma & \quad = -1 - \alpha - \delta - \alpha\delta - \frac{5}{6}\alpha^2 - \frac{5}{6}\delta^2 - \frac{11}{6}\beta\gamma + 0(2) \end{cases}$$

Case 4.

$$\begin{cases} i_1 = \gamma \\ j_1 = \alpha \end{cases} \quad \left(\partial_\alpha \Lambda^{i_2 j_2} \partial_{j_2} \Lambda^{\gamma\alpha} \partial_\gamma a \partial_{i_2} d\right)_{i_2 j_2=\alpha,\beta,\gamma,\delta} = 0(2)$$

Case 5.

$$\begin{cases} i_1 = \beta \\ j_1 = \delta \end{cases} \quad \left(\partial_\delta \Lambda^{i_2 j_2} \partial_{j_2} \Lambda^{\beta\delta} \partial_\beta a \partial_{i_2} d\right)_{i_2 j_2=\alpha,\beta,\gamma,\delta} = 0(2)$$

Case 6.

$$\begin{cases} i_1 = \delta \\ j_1 = \beta \end{cases} \quad \left(\partial_\beta \Lambda^{i_2 j_2} \partial_{j_2} \Lambda^{\delta\beta} \partial_\delta a \partial_{i_2} d\right)_{i_2 j_2=\alpha,\beta,\gamma,\delta} = \frac{1}{6}\beta\gamma + 0(2)$$

Case 7.

$$\begin{cases} i_1 = \gamma \\ j_1 = \delta \end{cases} \quad \left(\partial_\delta \Lambda^{i_2 j_2} \partial_{j_2} \Lambda^{\gamma\delta} \partial_\gamma a \partial_{i_2} d\right)_{i_2 j_2=\alpha,\beta,\gamma,\delta} = 0(2)$$

Case 8.

$$\begin{cases} i_1 = \delta \\ j_1 = \gamma \end{cases} \quad \left(\partial_\gamma \Lambda^{i_2 j_2} \partial_{j_2} \Lambda^{\delta\gamma} \partial_\delta a \partial_{i_2} d\right)_{i_2 j_2=\alpha,\beta,\gamma,\delta} = \frac{1}{6}\beta\gamma + 0(2)$$

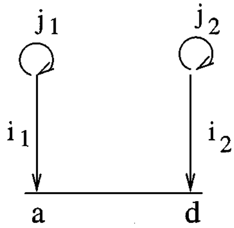


Fig. 4. Γ_2.

Case 9.

$$\begin{cases} i_1 = \alpha \\ j_1 = \delta \end{cases} \quad \left(\partial_\delta \Lambda^{i_2 j_2} \partial_{j_2} \Lambda^{\alpha\delta} \partial_\alpha a \partial_{i_2} d\right)_{i_2 j_2 = \alpha,\beta,\gamma,\delta} = 0(2)$$

Case 10.

$$\begin{cases} i_1 = \delta \\ j_1 = \alpha \end{cases} \quad \left(\partial_\alpha \Lambda^{i_2 j_2} \partial_{j_2} \Lambda^{\delta\alpha} \partial_\delta a \partial_{i_2} d\right)_{i_2 j_2 = \alpha,\beta,\gamma,\delta} = 0(2)$$

then we have

$$B_{\Gamma_1}(a, d) = -2 - 2\alpha - 2\delta - 2\alpha\delta - \frac{5}{3}\alpha^2 - \frac{5}{3}\delta^2 - \frac{10}{3}\beta\gamma + 0(2).$$

Similarly we calculate (see Fig. 4)

$$B_{\Gamma_2}(a, d) = \sum_{i_1 i_2 j_1 j_2} \partial_{j_2} \Lambda^{i_2 j_2} \partial_{j_1} \Lambda^{i_1 j_1} \partial_{i_1} a \partial_{i_2} d$$

thus we have

$$B_{\Gamma_2}(a, d) = -4 - 4\alpha - 4\delta - 4\alpha\delta - \frac{10}{3}\alpha^2 - \frac{10}{3}\delta^2 - \frac{32}{3}\beta\gamma + 0(2)$$

and for (see Fig. 5)

$$B_{\Gamma_3}(a, d) = \sum_{i_1 i_2 j_1 j_2} \partial_{j_2} \partial_{j_1} \Lambda^{i_2 j_2} \Lambda^{i_1 j_1} \left(\partial_{i_1} a \partial_{i_2} d + \partial_{i_2} a \partial_{i_1} d\right)$$

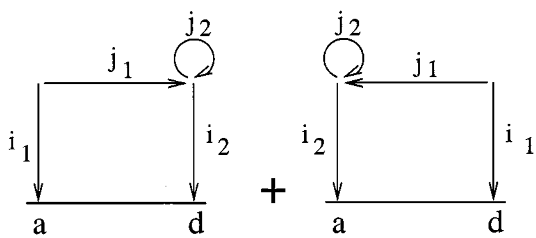


Fig. 5. Γ_3.

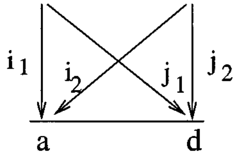


Fig. 6. Γ_4.

then we have

$$B_{\Gamma_3}(a, d) = -\frac{28}{3}\beta\gamma + 0(2)$$

and (see Fig. 6)

$$B_{\Gamma_4}(a, d) = \sum_{i_1 i_2 j_1 j_2} \Lambda^{i_2 j_2} \Lambda^{i_1 j_1} \partial_{i_2} \partial_{i_1} a \partial_{j_2} \partial_{j_1} d$$
$$B_{\Gamma_4}(a, d) = 2\beta\gamma + 0(2)$$

and (see Fig. 7)

$$B_{\Gamma_5}(a, d) = \sum_{i_1 i_2 j_1 j_2} \partial_{j_1} \Lambda^{i_2 j_2} \Lambda^{i_1 j_1} \left(\partial_{j_2} a \partial_{i_2} \partial_{i_1} d + \partial_{i_2} \partial_{i_1} a \partial_{j_2} d\right)$$
$$B_{\Gamma_5}(a, d) = \frac{8}{3}\beta\gamma + 0(2)$$

and (see Fig. 8)

$$B_{\Gamma_6}(a, d) = \sum_{i_1 i_2 j_1 j_2} \Lambda^{i_2 j_2} \partial_{j_1} \Lambda^{i_1 j_1} \left(\partial_{i_2} \partial_{i_1} a \partial_{j_2} d + \partial_{j_2} a \partial_{i_2} \partial_{i_1} d\right)$$
$$B_{\Gamma_6}(a, d) = \frac{16}{3}\beta\gamma + 0(2).$$

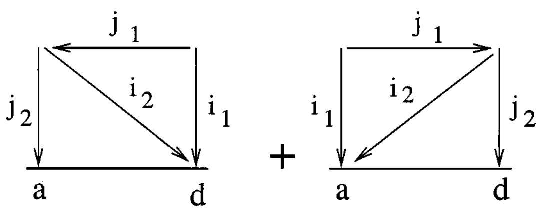


Fig. 7. Γ_5.

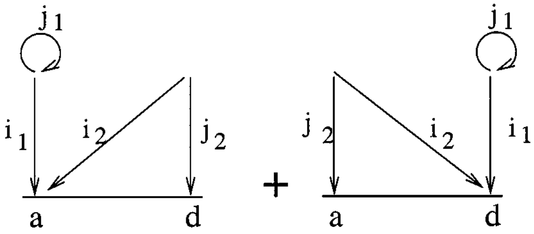


Fig. 8. Γ_6.

Now considering weights $a_{\Gamma_1}, a_{\Gamma_2}, a_{\Gamma_3}, a_{\Gamma_4}, a_{\Gamma_5}$, and a_{Γ_6}, we have

$$\begin{aligned} C_K(a,d) = {} & \left(-2a_{\Gamma_1} - 4a_{\Gamma_2}\right) + \left(-2a_{\Gamma_1} - 4a_{\Gamma_2}\right)\alpha + \left(-2a_{\Gamma_1} - 4a_{\Gamma_2}\right)\delta \\ & + \left(-2a_{\Gamma_1} - 4a_{\Gamma_2}\right)\alpha\delta + \left(-\frac{5}{3}a_{\Gamma_1} - \frac{10}{3}a_{\Gamma_2}\right)\alpha^2 \\ & + \left(-\frac{5}{3}a_{\Gamma_1} - \frac{10}{3}a_{\Gamma_2}\right)\delta^2 + \left(-\frac{10}{3}a_{\Gamma_1} - \frac{32}{3}a_{\Gamma_2} - \frac{28}{3}a_{\Gamma_3}\right. \\ & \left. + 2a_{\Gamma_4} + \frac{8}{3}a_{\Gamma_5} + \frac{16}{3}a_{\Gamma_6}\right)\beta\gamma. \end{aligned}$$

In the other hand, we have

$$C_T(a,d) = 0$$

then we obtain our two first equations. In the same way we obtain the remaining equations.

Proposition 3 (*Comparing on exponential chart*). *In the exponential chart we can't write the Takhtajan star product as a K-star product.*

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REFERENCES

Arnal, D., Ben Amar, N., and Masmoudi, M. (1999). Cohomology of good graphs and Kontsevich's star-product. *Letters in Mathematical Physics* **48**, 291–306.

Arnal, D. and Masmoudi. M. (2002). *Cohomologie de Hochschild des graphes de Kontsevich, Bull. Soc. Math. France.*

Dito, G. (1999). Kontsevich star product on the dual of a Lie algebra. *Letters in Mathematical Physics* **48**, 307–322.

Drinfeld, V. G. (1986). Quantum groups. In *Proceedings of International Congress of Mathematicians*, Berkely, 1986, Vol. 1. p. 798.

Gutt, S. (1983). An explicit $*$-product on the cotangent bundle of a Lie group. *Letters in Mathematical Physics* **7**, 249–258.

Kontsevich, M. Deformation quantization of Poisson manifolds, Preprint q-alg/9709040.

Kathotia, V. Kontsevich's universal formula for deformation quantization and the Campbell–Baker–Hausdorff formula I, Preprint q-alg/9811174.

Shoikhet, B. On the Kontsevich and the Campbell–Baker–Hausdorff deformation quantization of a linear Poisson structure, Preprint q-alg/9903036.

Takhtajan, L. A. (1989). Lectures on Quantum Groups, World scientific, Singapore.